Chapter 1

The Physics of Magnetism

Suggested Supplemental Reading

For background:

pp 1-4: Butler (1992)

Chapters on magnetism from your favorite college physics book for review.

To learn more:

Chapter 1: Jiles (1992)

Chaper 1: Cullity (1972)

In this lecture we will review the basic physical principles behind magnetism. We will be using primarily the Système International (SI) units that are based on meters-kilograms-seconds. There are other sytems of units that are important in magnetism and the most prevalent of these (electromagnetic units of the cgs system) will be covered later in the lecture.

1.1 What is a magnetic field?

Magnetic fields, like gravitational fields, cannot be seen or touched. We can feel the pull of the Earth's gravitational field on ourselves and the objects around us, but we do not experience magnetic fields in such a direct way. We know of the existence of magnetic fields by their effect on objects such as magnetized pieces of metal, naturally magnetic rocks such as lodestone, or temporary magnets such as copper coils that carry an electrical current. If we place a magnetized needle on a cork in a bucket of water, it will slowly align itself with the local magnetic field. Turning on the current in a copper wire can make a nearby compass needle jump. Observations like these led to the development of the concept of magnetic fields.

Electric currents make magnetic fields, so we can define what is meant by a "magnetic field" in terms of the electric current that generates it. Figure 1.1a is a picture of what happens when we pierce a flat sheet with a wire carrying a current i. When iron filings are sprinkled on the sheet, the filings line up with the magnetic field produced by the current in the wire. A circle tangential to the field is shown in Figure 1.1b, which illustrates the *right-handrule* (see inset to Figure 1.1b). If your right thumb points in the direction of (positive) current flow (the direction opposite the flow of the electrons), your fingers will curl in the direction of the magnetic field.

The magnetic field $\mathbf{H}$ points at right angles to both the direction of current flow and to the radial unit vector $\mathbf{r}$ in Figure 1.1b. The magnitude of $\mathbf{H}$ is proportional to the strength of the current i. In the simple case illustrated in Figure 1.1b the magnitude of $\mathbf{H}$ is given by Ampère's

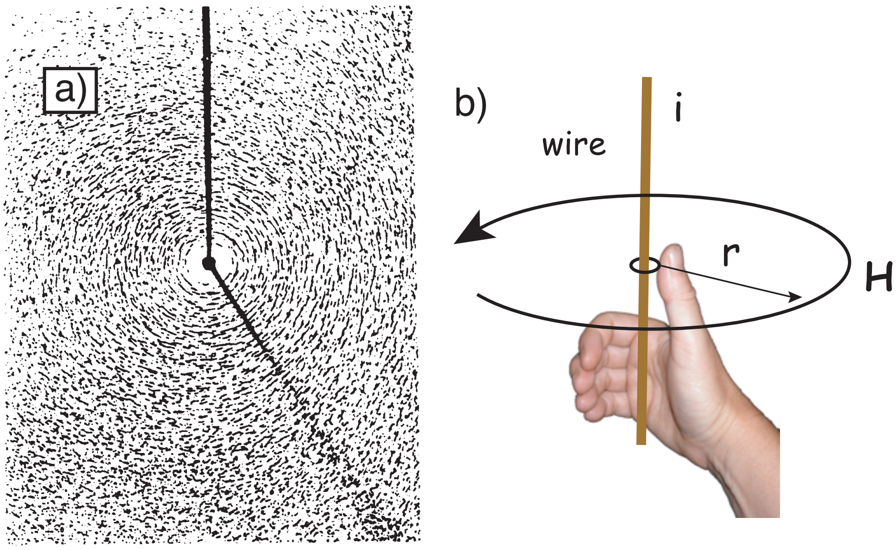

Figure 1.1: a) Distribution of iron filings on a flat sheet pierced by a wire carrying a current i. b) Relationship of magnetic field to current for straight wire. [Iron filings picture from Jiles (1992).]

law:

$$H = \frac{i}{2\pi r}.$$

So, now we know the units of $\mathbf{H}$: Am^{-1}.

Ampère's Law in its most general form is one of Maxwell's equations of electromagnetism: In a steady electrical field, $\nabla \times \mathbf{H} = \mathbf{J}_f$, where $\mathbf{J}_f$ is the electric current density. In other words: The curl (or circulation) of the magnetic field is equal to the current density. The origin of the term "curl" for the cross product of the gradient operator with a vector field is suggested in Figure 1.1a in which the iron filings seem to curl around the wire.

1.2 Magnetic moment

We have seen that an electrical current in a wire produces a magnetic field that curls around the wire. If we bend the wire into a loop with an area πr^2 that carries a current i, as shown in Figure 1.2a, the current loop creates the magnetic field shown by pattern of the the iron filings. This magnetic field is that same as the field that would be produced by a magnet with a magnetic moment $\mathbf{m}$ shown in Figure 1.2b. This moment is created by the current i and also depends on the area of the current loop (the bigger the loop, the bigger the moment), hence $\mathbf{m} = i\pi r^2$. The moment created by a set of loops (as shown in Figure 1.2c is the sum of the n individual loops, i.e.:

$$\mathbf{m} = ni\pi r^2. \tag{1.1}$$

So, now we know the units of $\mathbf{m}$: Am^2.

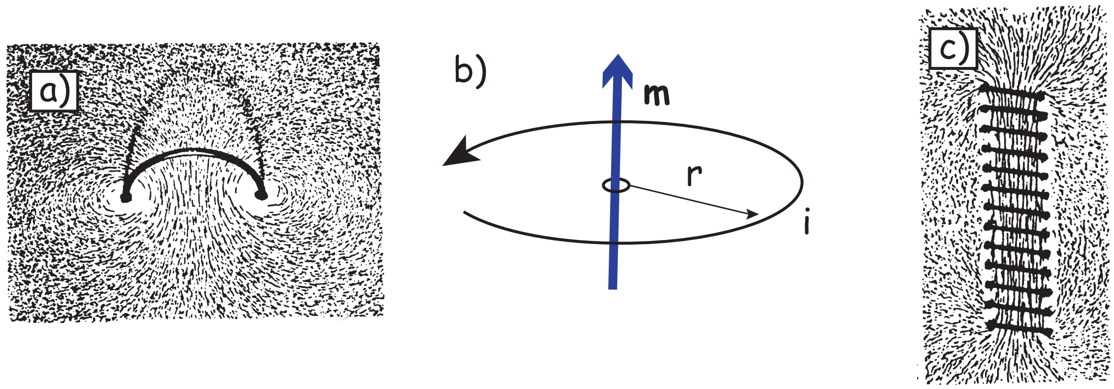

Figure 1.2: a) Iron filings show the magnetic field generated by current flowing in a loop. b) The magnetic field of a current loop with current i and area πr^2 is the same as one produced by a magnet with moment $\mathbf{m}$. c) The magnetic field of loops arranged as a solinoid is the sum of the contribution of the individual loops. [Iron filings pictures from Jiles (1992).]

1.3 Magnetic flux

The magnetic field is a vector field because at any point the field has both direction and magnitude. Consider the field of a bar magnet made visible by iron filings as shown in Figure 1.3. The direction of the field at any point is given by the arrows while the strength depends on how close the field lines are to one another. The magnetic field lines are known as "magnetic flux". The density of flux lines is one measure of the strength of the magnetic field: the magnetic induction $\mathbf{B}$.

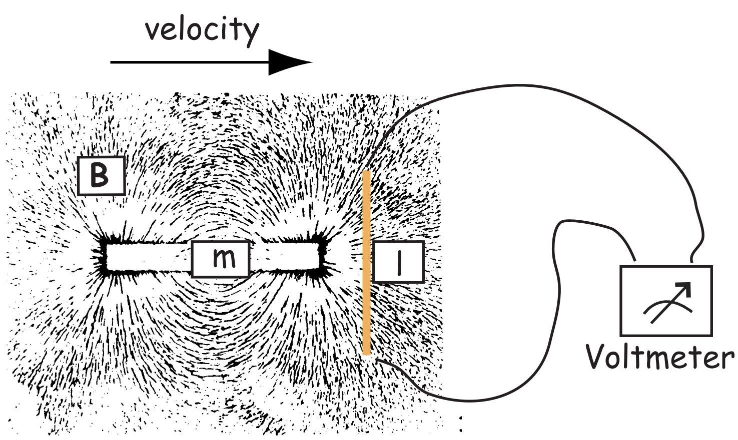

Figure 1.3: A magnetic moment $\mathbf{m}$ makes a vector field $\mathbf{B}$ made visible by the iron filings. If this field moves with velocity $\mathbf{v}$, it generates a voltage V in an electrical conductor of length l. [Iron filings picture from Jiles (1992).]

Magnetic flux density (i.e., magnetic induction) can therefore be quantified when a conductor moves through it. Magnetic induction can be thought of as something that creates a potential difference with voltage V in a conductor of length l when the conductor moves relative to the magnetic induction B with velocity $\mathbf{v}$ (see Figure 1.3): $V = vlB$. From this we can derive the units of magnetic induction: the tesla (T). One tesla is the magnetic induction that generates a potential of one volt in a conductor of length 1 meter when moving 1 meter per second. Ergo, 1 T = 1 V $\cdot$ s $\cdot$ m^{-2}.

Another way of looking at this is that if magnetic induction ($\mathbf{B}$) is the flux density, this must be the flux Φ per unit area. So an increment of flux $d\Phi$ is the field B times the increment of area dA. The area here is the length of the wire l times its displacement ds in time dt. The instantaneous velocity is $dv = ds/dt$ so or $d\Phi = BdA$ and the rate of change of flux is:

$$\frac{d\Phi}{dt} = (\frac{ds}{dt})Bl = vBl = V. \tag{1.2}$$

Equation 1.2 is known as Faraday's law and in its most general form is the fourth of Maxwell's equations. We see from this equation that the units of magnetic flux must be a volt-second which is unit in its own right, the weber (Wb). The weber is defined as the amount of magnetic flux which, when passed through a one-turn coil of a conductor carrying a current of one ampere, produces an electric potential of one volt. This definition suggests a means to measure the strength of magnetic induction and is the basis of the "fluxgate" magnetometer.

1.4 Magnetic energy

A magnetic moment $\mathbf{m}$ has a *magnetostatic energy* (E_m) associated with it. This is the energy that tends to align compass needles with the magnetic field (see Figure 1.4. This energy is given by $\mathbf{m} \cdot \mathbf{B}$ or $mB\cos\theta$ where m and B are the magnitudes of $\mathbf{m}$ and $\mathbf{B}$, respectively. Magnetic energy has of joules.

1.5 Magnetization and magnetic susceptibility

Magnetization $\mathbf{M}$ is a moment per unit volume (units of Am^{-1}) or per unit mass (Am^{2}kg^{-1}). Subatomic charges such as protons and electrons can be thought of as tracing out tiny circuits and behaving as tiny magnetic moments. They respond to external magnetic fields and give rise to an induced magnetization. The relationship between the magnetization induced in a material $\mathbf{M}_I$ and the external field $\mathbf{H}$ is defined as:

$$\mathbf{M}_I = \chi_b \mathbf{H}. \tag{1.3}$$

The parameter χ_b is known as the *bulk magnetic susceptibility* of the material; it can be a complicated function of orientation, temperature, state of stress, time scale of observation and applied field but is often treated as a scalar.

Certain materials can produce magnetic fields in the absence of external magnetic fields (i.e., they are permanent magnets). As we shall see later in the course, these so-called "spontaneous" magnetic moments are also the result of spins of electrons which, in some crystals, act in a coordinated fashion, thereby producing a net magnetic field. The resulting magnetization can be fixed by various mechanisms and can preserve records of ancient magnetic fields. This *remanent magnetization* forms the basis of the field of paleomagnetism and will be discussed at length in the rest of this class.

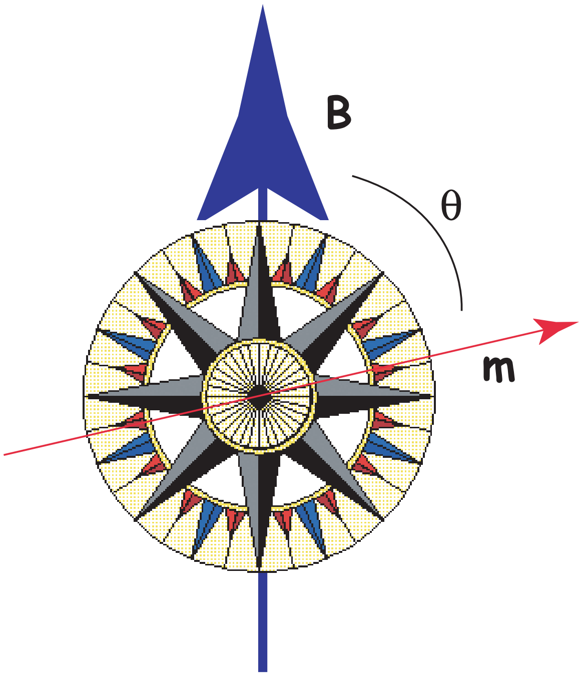

Figure 1.4: A magnetic moment **m** of for example a compass needle, will tend to align itself with a magnetic field **B**. The aligning energy is the magnetostatic energy which is greatest when the angle between the two vectors θ is at a maximum.

1.6 Relationship of B and H

From the foregoing discussion, we see that **B** and **H** are closely related. In paleomagnetic practice, both **B** and **H** are referred to as the "magnetic field". Strictly speaking, **B** is the induction and **H** is the field, but the distinction is often blurred. The relationship between **B** and **H** is given by:

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}). \tag{1.4}$$

where μ_o is a parameter known as "the permeability of free space". In the SI system, μ_o has dimensions of henries per meter and is given by $\mu_o = 4\pi \times 10^{-7}\mathrm{H} \cdot \mathrm{m}^{-1}$.

1.7 A brief tour of magnetic units in the cgs system

So far, we have derived magnetic units in terms of the Système International (SI). In practice, you will notice that in many laboratories and in the literature people frequently use what are known as cgs units, based on centimeters, grams and seconds. You may wonder why any fuss would be made over using meters as opposed to centimeters because the conversion is trivial. With magnetic units, however, the conversion is far from trivial and has been the source of confusion and many errors. So, in the interest of clearing things up, we will briefly outline the cgs approach to magnetic units.

The derivation of magnetic units in cgs is entirely different from SI. The approach we will take here (see Cullity, 1972) starts with the concept of a magnetic pole with strength p. By analogy to Coulomb's law, the force between two poles p_1, p_2 instead of with current loops as we did for SI units. Coulomb's Law states that the force between two charges (q_1, q_2) is:

$$F_{12} = k\frac{q_1 q_2}{r^2} \tag{1.5}$$

where r is the distance between the two charges. In cgs units, the proportionality constant k is simply unity, whereas in SI units it is $\frac{1}{4\pi\epsilon_0}$ where $\epsilon_0 = \frac{10^7}{4\pi C^2}$ and c is the speed of light in a vacuum (hence $\epsilon_0 = 8.859 \cdot 10^{-12}$ AsV^{-1}m^{-1}). [You can see why many people really prefer cgs but we are not allowed to publish in cgs in AGU journals so we just must grin and bear it!]

For magnetic units, we use pole strength p_1, p_1 in units of "electrostatic units" or esu, so Equation 1.5 becomes

$$F = \frac{p_1 p_2}{r^2}$$

Force in cgs is in units of dynes (dyn) so,

$$F = 1\text{dyn} = \frac{1\text{g cm}}{s^2} = \frac{1\text{ esu}^2}{cm^2}$$

so 1 unit of pole strength is rather awkwardly 1 gm$^{1/2}$ cm$^{3/2}$ s^{-1}. Of course there are no isolated magnetic poles in nature, only dipoles, but the concept of a unit of pole strength lies at the heart of cgs magnetic units.

A magnetic pole, as an isolated electric charge, will create a magnetic induction $\mu_o H$ in the space around it. One unit of field strength (defined as one "oersted" or Oe) is the unit of field strength that exerts a force of one dyne on a unit of pole strength. The relationship between force, pole and field is written as:

$$F = p\mu_o H.$$

So, a pole with one pole strength, placed in a one Oe field is acted on by a force of one dyne. This is the same force that it would experience if placed one centimeter away from another pole with one pole strength. Hence, the field of this monopole must be one oersted at one centimeter away, and fall off as $1/r^2$.

Returning to the lines of force idea developed for magnetic fields earlier, let us define the oersted to be 1 line of force per square centimeter. Imagine a sphere with a radius r surrounding the magnetic monopole. The surface area of such a sphere is $4\pi r^2$. The sphere is a unit sphere ($r = 1$), the field strength at the surface is 1 Oe, then there must be 4π lines of force passing through it.

Proceeding to the notion of magnetic moment, from a cgs point of view, we start with a magnet of length l with two poles of strength p at each end. Placing the magnet in a field $\mu_o \mathbf{H}$, we find that it experiences a torque Γ proportional to $p, l, \mathbf{H}$ such that

$$\Gamma = pl \times \mu_o \mathbf{H}. \tag{1.6}$$

Recalling our earlier discussion of magnetic moment, you will realize that pl is simply the magnetic moment m. The units of torque are energy, which are ergs in cgs, so the units of magnetic moment are ergs/oersted. We therefore define the "electromagnetic unit" (emu) as being one erg/oersted. [Some use emu to refer to the magnetization (volume normalized moment, see above), but this is incorrect.]

You will have noticed the use of the parameter μ_0 in the above treatment - a parameter missing in Cullity (1972) and in many books and articles using the cgs units. The reason for this is that μ_0 is unity in cgs units and simply converts from oersteds (**H**) and gauss (**B**) which are therefore used interchangeably. It was inserted in this derivation to remind us that there IS a difference and that the difference becomes very important when we convert to SI because μ_0 is not unity, but 4π x 10^{-7}! For conversion between commonly used cgs and SI paramters, please refer to Table 1. 1.

Table 1.1: Conversion between SI and cgs units.

Parameter	SI unit	cgs unit	Conversion
Magnetic moment (**m**)	Am^2	emu	$1 \text{ A m}^2 = 10^3$ emu
Magnetization (**M**)	Am^{-1}	emu cm^{-3}	$1 \text{ Am}^{-1} = 10^{-3}$ emu cm^{-3}
Magnetic Field (**H**)	Am^{-1}	Oersted (oe)	$1 \text{ Am}^{-1} = 4\pi$ x 10^{-3} oe
Magnetic Induction (**B**)	T	Gauss (G)	$1 \text{ T} = 10^4$ G
Permeability			
of free space (μ_0)	Hm^{-1}	1	4π x 10^{-7} $Hm^{-1} = 1$
Susceptibility (χ)			
total ($\frac{\mathbf{m}}{\mathbf{H}}$)	m^3	emu oe^{-1}	$1 \text{ m}^3 = \frac{10^6}{4\pi}$ emu oe^{-1}
by volume ($\frac{\mathbf{M}}{\mathbf{H}}$)	-	emu cm^{-3} oe^{-1}	1 S.I. $= \frac{1}{4\pi}$ emu cm^{-3} oe^{-1}
by mass ($\frac{\mathbf{m}}{m} \cdot \frac{1}{\mathbf{H}}$)	m^3kg^{-1}	emu g^{-1} oe^{-1}	$1 \text{ m}^3\text{kg}^{-1} = \frac{10^3}{4\pi}$emu g^{-1} oe^{-1}

$1 \text{ H} = \text{kg m}^2\text{A}^{-2}\text{s}^{-2}$, $1 \text{ emu} = 1 \text{ G cm}^3$, $B = \mu_o(H + M)$, $1 \text{ T} = \text{kg A}^{-1} \text{ s}^{-2}$

1.8 The magnetic potential

An isolated electrical charge produces electrical fields that begin at the source (the charge) and spread (diverge) outward (see Figure 1.5a). Because there is no return flux to an oppositely charged "sink", there is a net flux out of the dashed box shown in the figure. The "divergence" of the electrical field is defined as $\nabla \cdot \mathbf{E}$ which quantifies the net flux (see supplement to Chapter 1 for more). In the case of the field around an electric charge, the divergence is non-zero.

Magnetic fields are different from electrical fields in that there is no equivalent to an isolated electrical charge; there are only pairs of "opposite charges", or magnetic *dipoles*. Therefore, any line of flux starting at one magnetic pole, returns to its sister pole and there is no net flux out of the box shown in Figure 1.5b; the magnetic field has no divergence (Figure 1.5b). This property of magnetic fields is another of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$.

We have already seen that the curl of the magnetic field ($\nabla \times \mathbf{H}$) depends on the current density which is not always zero. Therefore, magnetic fields cannot generally be represented as the gradient of a scalar field. However, in the special case away from electric currents, the magnetic field can be written as the gradient of a scalar field that is known as the *magnetic potential* ψ_m, *i.e.*,

$$\mathbf{H} = -\nabla\psi_m.$$

The presence of a magnetic moment **m** creates a magnetic field which is the gradient of a scalar field. We also know that the divergence of the magnetic field is zero, hence $\nabla^2\psi_m = 0$. This is LaPlace's equation which is the starting point for spherical harmonic analysis discussed briefly in Lecture 2.

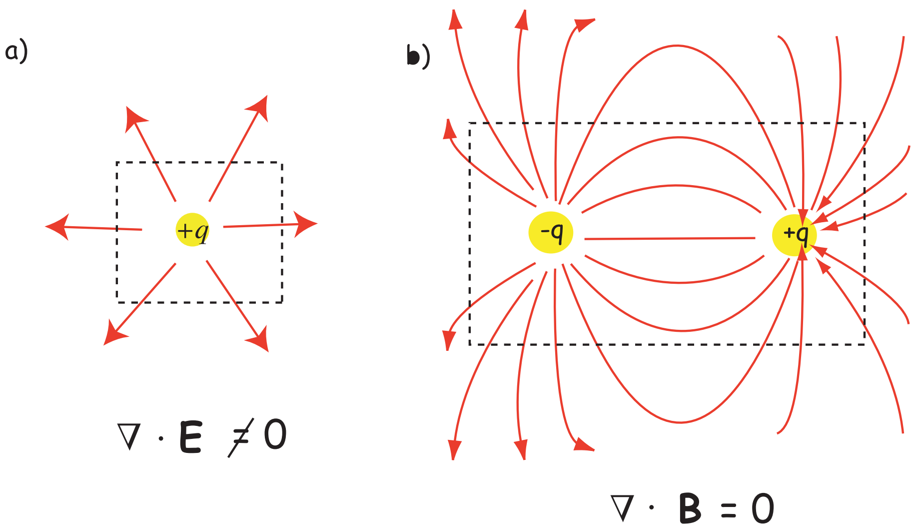

Figure 1.5: a) An electric charge produces a field that diverges out from the source. There is a net flux out of the dashed box, quantified by the divergence ($\nabla \cdot \mathbf{E}$), which is is proportional to the magnitude of the sources inside the box. b) there are no isolated magnetic charges, only dipoles. Within any space (e.g., the dashed box) any flux line that comes in, goes out. The divergence of such a field is zero, i.e., $\nabla \cdot \mathbf{B} = 0$.

The magnetic potential ψ_m is a function the vector $\mathbf{r}$ with radial distance r and angle θ from the moment. Given a *dipole moment* $\mathbf{m}$, the solution to LaPlace's equation for the simple case of a magnetic field produced by $\mathbf{m}$ is:

$$\psi_m = \frac{\mathbf{m} \cdot \mathbf{r}}{4\pi r^3} = \frac{m \cos\theta}{4\pi r^2}. \tag{1.7}$$

The radial and tangential components of $\mathbf{H}$ at P (Figure 1.6) are:

$$H_r = -\frac{\partial \psi_m}{\partial r} = \frac{1}{4\pi}\frac{2m\cos\theta}{r^3},$$

and

$$H_\theta = -\frac{1}{r}\frac{\partial V_m}{\partial \theta} = \frac{m \sin\theta}{4\pi r^3},$$

respectively.

1.9 The geodynamo

Maxwell's equations tell us that electric and changing magnetic fields are closely linked and can effect each other. Moving an electrical conductor through a magnetic field will cause electrons to flow, generating an electrical current. This is the principal of electric motors. In Figure 1.7 we see a design for a machine that will turn mechanical energy into magnetic field energy. The rotating disk is made of metal. As the disk turns in the presence of an initial magnetic field, the electrons scurry at right angles to the field, generating an electric potential (Figure 1.7b). The

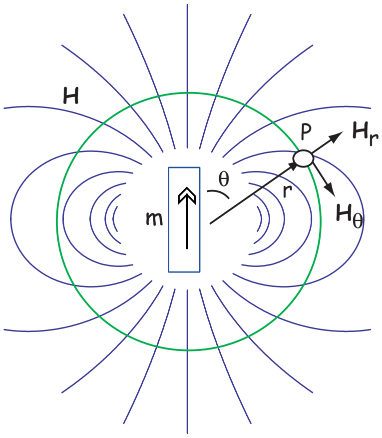

Figure 1.6: Field $\mathbf{H}$ produced at point P by a magnetic moment $\mathbf{m}$. $\mathbf{H}_r$ and $\mathbf{H}_\theta$ are the radial and tangential fields respectively.

brush connection allows a current to flow through the wire wound into a coil, in turn generating a magnetic field. If the rotating disk is spun in the right direction, the magnetic field will be in the same sense as the initial field, amplifying the effect and generating a much larger magnetic field. More complicated setups using two disks whose fields interact with one another generate chaotic magnetic behavior that can switch polarities even if the mechanical motion remains steady. While a very poor analogue for the Earth's magnetic field, it demonstrates that moving electrical conductors can generate a magnetic field. In the Earth of course the moving electrical conductor is the molten iron outer core.

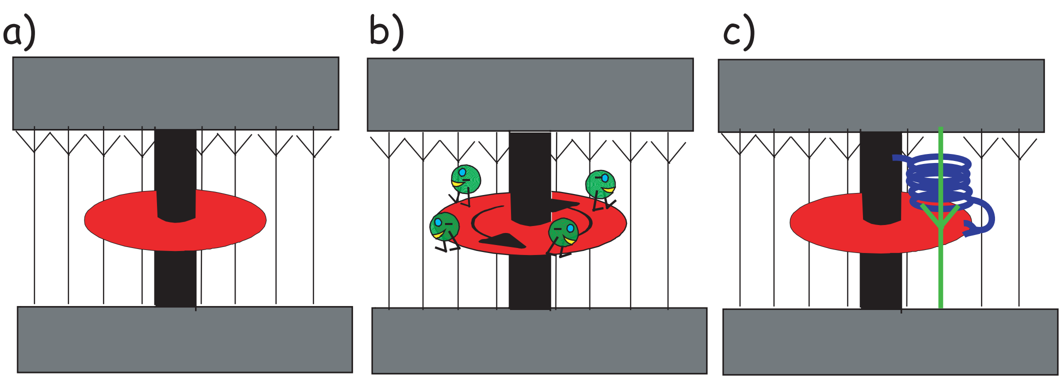

Figure 1.7: The Faraday disk dynamo. a) An initial field is produced by the electromagnet (thin arrows). The red disk is a conducting plate. b) When the conducting plate is rotated, electric charge moves perpendicular to the magnetic field setting up an electric potential between the inner conducting rod and the outer rim of the plate. c) When the conducting plate is connected to a coil wound such that a current produces a magnetic field in the same direction as the initial field, the magnetic field is enhanced. (Figure drawn with help from Philip Staudigel).

Appendix

In this appendix we will review the basic math concepts necessary to understand the chapter on magnetism. We will start with basic vector math and then review useful operators grad, div and curl.

A Vectors

A1 Addition

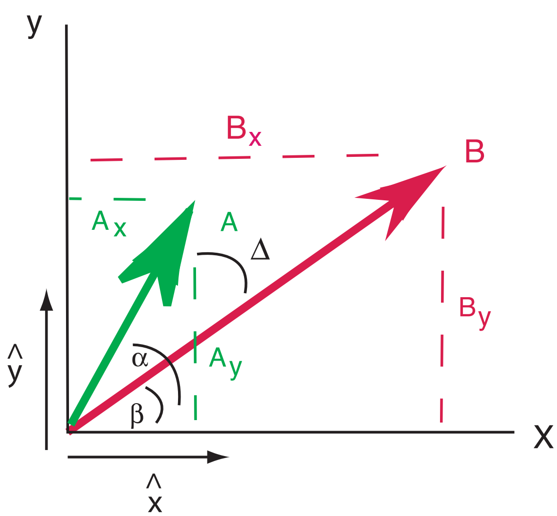

Figure A1: Vectors **A** and **B**, their components $A_{x,y}$, $B_{x,y}$ and the angles between them and the X axis, α and β. The angle between the two vectors is α -$\beta = \Delta$. Unit vectors in the directions of the axes are $\hat{x}$ and $\hat{y}$ respectively.

To add the two vectors (see Figure A1) **A** and **B**, we break each vector into components $A_{x,y}$ and $B_{x,y}$. For example, $A_x = |A| \cos\alpha, A_y = |A| \sin\alpha$ where $|A|$ is the length of the vector **A**. The components of the resultant vector **C** are: $C_x = A_x + B_x, C_y = A_y + B_y$. These can be converted back to polar coordinates of magnitude and angles if desired.

A2 Subtraction

To subtract two vectors, compute the components as in addition, but the components of the vector difference **C** are: $C_x = A_x - B_x, C_y = A_y - B_y$.

A3 Multiplication

There are two ways to multiply vectors. The first is the dot product whereby $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y$. This is a scalar and is actually the cosine of the angle between the two vectors if the **A** and **B** are taken as unit vectors (assume a magnitude of unity in the component calculation.

The other way to perform vector multiplication is the cross product (see Figure A2), which produces a vector orthogonal to both **A** and **B** and whose components are given by:

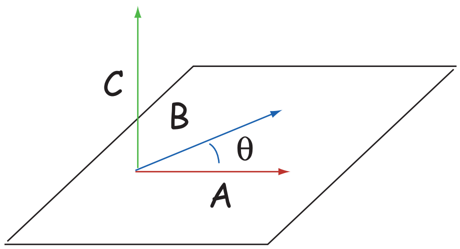

Figure A2: Illustration of cross product of vectors A and B separated by angle θ to get the orthogonal vector C.

$$C = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

To calculate the determinant, we follow these rules:

$$C_x = A_yB_z - A_zB_y, C_y = A_zB_x - A_xB_z, C_z = A_xB_y - A_yB_x.$$

or

$$C_i = A_jB_k - A_kB_j \qquad i \neq j \neq k$$

A4 Change of coordinates

In paleomagnetism, we often have to change coordinate systems, from say sample coordinates to geographic, or to correct for tilting of the geological units. The way to do this in a simple 2-D case is illustrated in Figure A3. Given the vector shown in Figure A3a, that is oriented at an angle α from the $\mathbf{X}_1$ axis. To change to a second set of axes $\mathbf{X}'_1, \mathbf{X}'_2$, we first have to define a set of coefficients called "direction cosines". For example, the direction cosine a_{12} is the cosine of the angle between the old X_1 and the new X'_2, α_{12}. We can define four of these direction cosines to fully describe the relationship between the two coordinate systems:

$$\begin{aligned} a_{11} &= \cos\alpha_{11}, a_{21} = \cos\alpha_{21}, \\ a_{12} &= \cos\alpha_{12}, a_{22} = \cos\alpha_{22}. \end{aligned}$$

The first subscript always refers to the old system and the second refers to the new.

To find the new coordinates x'_1, x'_2 from the old, we just have:

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2, \\ x'_2 &= a_{21}x_2 + a_{22}x_2. \end{aligned}$$

In three dimensions we have:

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x'_2 &= a_{21}x_2 + a_{22}x_2 + a_{23}x_3, \\ x'_3 &= a_{31}x_2 + a_{32}x_2 + a_{33}x_3, \end{aligned}$$

A short cut notation to this is: $x'_i = a_{ij}x_j$. This just means that for each axis i, just sum through the j's for all the dimensions.

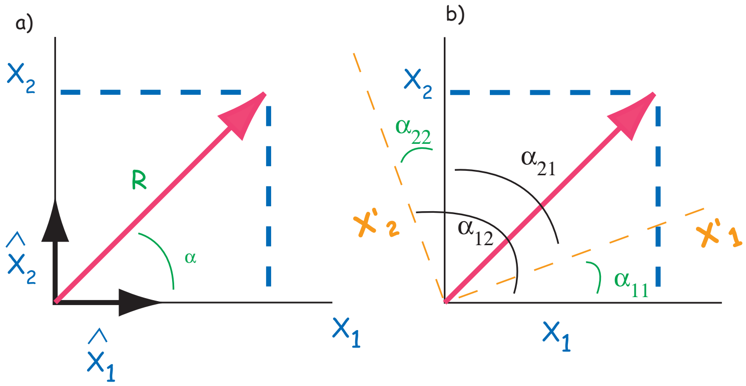

Figure A3: Transformation of axes. a) Definition of vector in one set of coordinates, x_1, x_2. b) Definition of angles relating old X axes to new X'.

B Upside down triangles

B1 Gradient, ∇

We often wish to differentiate a function along three orthogonal axes. For example, imagine we know the topography of a ski area (see Figure B1). For every location (in say, X and Y coordinates), we know the height above sea level. This is a scalar function. Now imagine we want to build a ski resort, so we need to know the direction of steepest descent and the slope (red arrows in Figure B1).

Figure B1: Illustration of the relationship between a vector field (direction and magnitude of steepest slope at every point, e.g., red arrows) and the scalar field (height) of a ski slope.

To convert the scalar field (height versus position) to a vector field (direction and magnitude of greatest slope) mathematically, we would simply differentiate the topography function. Let's say we had a very weird two dimensional, sinusoidal topography such that $z = f(x) = \sin x$ with z the height and x is the distance from some marker. The slope in the x direction ($\hat{x}$), then would be $\hat{x}\frac{d}{dx}f(x)$. If $f(x, y, z)$ were a three dimentional topography then the gradient of the topography function would be:

$$(\hat{x}\frac{\partial}{\partial x}f + \hat{y}\frac{\partial}{\partial y}f + \hat{z}\frac{\partial}{\partial z}f)$$

For short hand, we define a "vector differential operator" to be a vector whose components are

$$\nabla = (\hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z}).$$

This can also be written in polar coordinates:

$$\nabla = \frac{\partial}{\partial r}, \frac{\partial}{r\partial\theta}, \frac{\partial}{r\sin\theta\partial\phi}$$

Just as the direction and magnitude of maximum slope of the topography is a the gradient of the scalar function of height, the magnetic field is the gradient of a scalar function of something we will define as the magnetic potential. In Lecture 1, we said that the magnetic field $\mathbf{H}$ is the gradient of a scalar potential field ψ_m, or

$$\mathbf{H} = -\nabla\psi_m.$$

This means that for a simple dipolar field:

$$\psi_m = \frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3}$$

We can derive the radial component of the field as:

$$H_r = \frac{\partial\psi_m}{\partial r} = \frac{1}{4\pi}\frac{2m\cos\theta}{r^3}$$

and the tangential component as:

$$H_\theta = \frac{-1}{r}\frac{\partial V_m}{\partial\theta} = \frac{m\sin\theta}{4\pi r^3}.$$

B2 Divergence

The divergence of a vector function (e.g. $\mathbf{H}$) is written as:

$$\nabla\cdot\mathbf{H}$$

The trick here is to treat ∇ as a vector and use the rules for dot products described in the section A of this appendix. In cartesian coordinates, this is:

$$\nabla\cdot\mathbf{H} = \hat{x}\frac{\partial H_x}{\partial x} + \hat{y}\frac{\partial H_y}{\partial y} + \hat{z}\frac{\partial H_z}{\partial z}.$$

Like all dot products, the divergence of a vector function is a scalar.

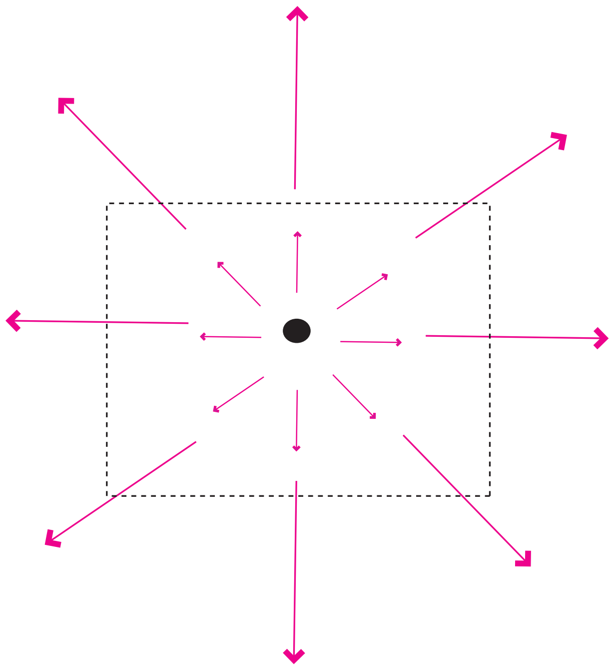

Figure B2: Example of a vector field with a non-zero divergence.

B3 Geometrical interpretation of divergence

The name divergence is well chosen because $\nabla \cdot \mathbf{H}$ is a measure of how much the vector field "spreads out" (diverges) from the point in question. In fact, what divergence quantifies is the balance between vectors coming in to a particular region versus those that go out. The example in Figure B2 depicts a vector function whereby the magnitude of the vector increases linearly with distance away from the central point. An example of such a function would be $v(r) = r$. The divergence of this function is:

$$\nabla \cdot v = \frac{\partial}{\partial r} r = 1$$

(a scalar). There are no arrows returning in to the dashed box, only vectors going out and the non-zero divergence quantifies this net flux out of the box.

Now consider Figure B3, which depicts a vector function that is constant over space, i.e. $v(r) = k$. The divergence of this function is:

$$\nabla \cdot v = \frac{\partial}{\partial r} k = 0.$$

The zero divergence means that for every vector leaving the box, there is an equal and opposite vector coming in. Put another way, no net flux results in a zero divergence. The fact that the divergence of the magnetic field is zero means that there are no point sources (monopoles), as opposed to electrical fields that have divergence related to the presence of electrons or protons.

B4 Curl

The curl of the vector function $\mathbf{B}$ is defined as $\nabla \times \mathbf{B}$. In cartesian coordinates we have

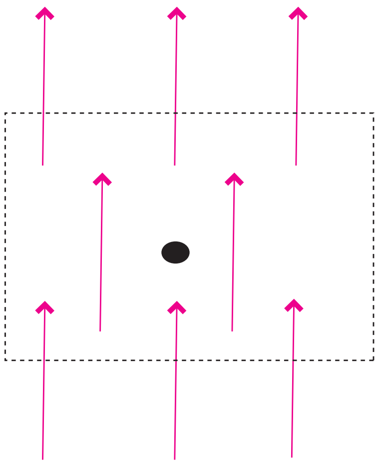

Figure B3: Example of a vector field with zero divergence.

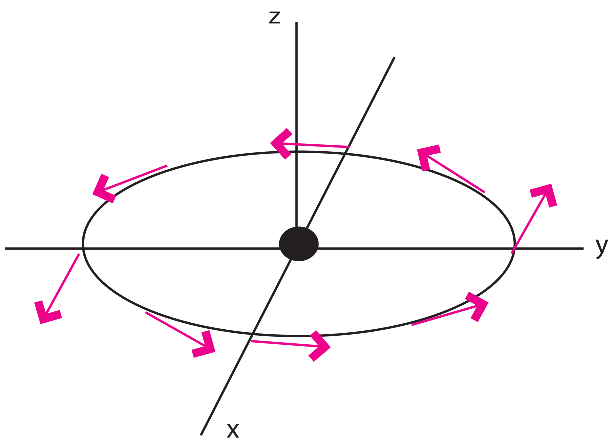

Figure B4: Example of a vector field with non-zero curl.

$$\nabla \times \mathbf{B} = \hat{x}(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y) + \hat{y}(\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z) + \hat{z}(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x).$$

Curl is a measure of how much the vector function "curls" around a given point. The function describing the velocity of water in a whirlpool has a significant curl, while that of a smoothly flowing stream does not.

Consider Figure B4 which depicts a vector function $v = -y\hat{x} + x\hat{y}$. The curl of this function is:

$$\nabla \times \ v = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}.$$

or

$$\hat{x}(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x) + \hat{y}(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}(-y)) + \hat{z}(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y))$$
$$= 0\hat{x} + 0\hat{y} + 2\hat{z}$$

So there is a positive curl in this function and the curl is a vector in the $\hat{z}$ direction.

The magnetic field has a non-zero curl in the presence of currents or changing electric fields. In free space, away from currents (lightning!!), the magnetic field has zero curl.