Lectures in Paleomagnetism, 2005
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Citation:
http://earthref.org/MAGIC/books/Tauxe/2005/

June 8, 2005

Chapter 11

Fisher statistics for paleomagnetic directions

Suggested Reading

Background:
Chapters 1-5: Taylor (1982)
Chapter 6: Butler (1992)
To learn more:
Chapters 2-5: Fisher et al. (1987)

11.1 Introduction

In the previous few lectures we learned about routine paleomagnetic sampling and laboratory procedures. Once paleomagnetic directions have been obtained after stepwise demagnetization, principal component analysis, etc., one may wish to interpret them in terms of ancient geomagnetic field directions. To do this, there must be some way of calculating mean vectors and of quantifying the confidence intervals. But before we can understand the statistics of vectors we need some idea about statistics in general.

11.2 Statistics of scalars

The starting point for most statistical discussions is the so-called "normal" distribution, or "Gaussian" distribution. Let's say that we made 1000 measurements of some parameter, say bed thickness in a particular sedimentary formation in centimeters. We plot these in histogram form in Figure 11.1a.

A normal distribution can be characterized by two parameters, the mean (μ) and the variance σ^2. How to estimate the parameters of the underlying distribution is the art of statistics. We all know that the arithmetic mean of a batch of data $\bar{x}$ drawn from a normal distribution is calculated by:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

The mean estimated from the data shown in Figure 11.1 is 10.09. If we had measured an infinite number of bed thicknesses, we would have gotten the bell curve shown in the figure and calculated a mean of 10.

The "spread" in the data is characterized by the variance σ^2. Variance for normal distributions can be estimated by the parameter s^2:

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2.$$

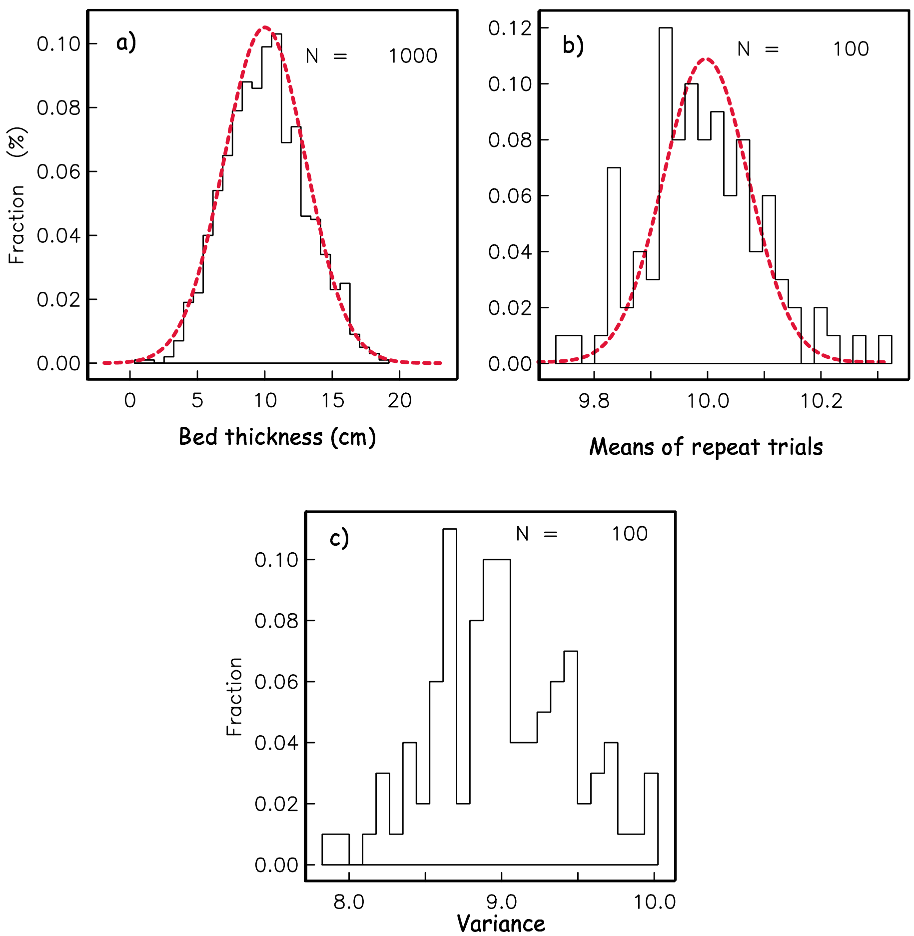

Figure 11.1: a) Histogram of 1000 measurements of bed thickness in some sedimentary formation. Also shown is the smooth curve of a normal distribution with a mean of 10 and a standard deviation of 3. b) Histogram of the means from 100 repeated sets of 1000 measurements from the same sedimentary formation. The distribution of the means is much tighter. c) Histogram of the variances (s^2) from the same set of experiments as in b). The distribution of variances is not bell shaped; it is χ^2.

In order to get the units right on the uncertainty in the mean (cm - not cm^2), we have to take the square root of s^2. The parameter s gives an estimate of the standard deviation σ and is the bounds around the mean that includes 63% of the values. The 95% confidence bounds are given by

1.96s (this is what a "2-σ error" is), and should include 95% of the means. The bell curve shown in Figure 11.1 has a σ (standard deviation) of 3, while the s is 2.97.

If you repeat the bed measuring experiment a few times, you will never get exactly the same measurements in the different trials. The mean and standard deviations measured for each trial then are "sample" means and standard deviations. If you plotted up all those sample means, you would get another normal distribution whose mean should be pretty close to the true mean, but with a much more narrow standard deviation. In Figure 11.1b we plot a histogram of means from 100 such trials of 1000 measurements each drawn from the same distribution of $\mu = 10, \sigma = 3$. In general, we expect the standard deviation of the means (or "standard error of the mean" s_m) to be related to s by

$$s_m = \frac{s}{\sqrt{N_{trials}}}$$

What if we were to plot up a histogram of the estimated variances as in Figure 11.1c? Are these also normally distributed? The answer is no, because variance is a squared parameter relative to the original units. In fact, the distribution of variance estimates from normal distibutions is expected to be "chi-squared" (χ^2). The width of the χ^2 distribution is also governed by how many measurements were made. The so-called "number of degrees of freedon" ν is given by the number of measurements made minus the number of measurements required to make the estimate, so ν for our case is $N-1$. Therefore we expect the variance estimates to follow a χ^2 distribution with $N-1$ degrees of freedom of χ^2_ν.

We often wish to consider ratios of variances (for example to decide if the data are more scattered in one data set relative to another). In order to do this, we must know what ratio would be expected from data sets drawn from the same distributions. Ratios of such variances follow a so-called F distribution with ν_1 and ν_2 degrees of freedom for the two data sets. This is denoted $F[\nu_1, \nu_2]$. Thus if the ratio f, given by:

$$f = \frac{s_1^2}{s_2^2}$$

is greater than the 5% critical value of $F[\nu_1, \nu_2]$ (check in the F-table in the Appendix), the hypothesis that the two variances are the same can be rejected at the 95% level of confidence.

11.3 Statistics of vectors

We turn now to the trickier problem of sets of measured vectors. We will consider the case in which all vectors are assumed to have a length of one, i.e., these are unit vectors. Unit vectors are just "directions".

Consider the various sets of directions plotted as equal area projections (see Lecture 2) in Figure 11.2. These are all measurements of a single, vertical direction, but with varying degrees of precision. It would be handy to be able to calculate a mean direction for the data sets and to quantify the precision of the measurements.

The average inclination, calculated as the arithmetic mean of the inclinations, will obviously not be vertical. We will see, however, that the vector mean of the directions of each data set is actually nearly vertical as it should be. In the following, we will demonstrate the proper way to calculate mean directions and confidence regions for directional data that are distributed in the manner shown in Figure 11.2. We will also briefly describe several useful statistical tests that are popular in the paleomagnetic literature.

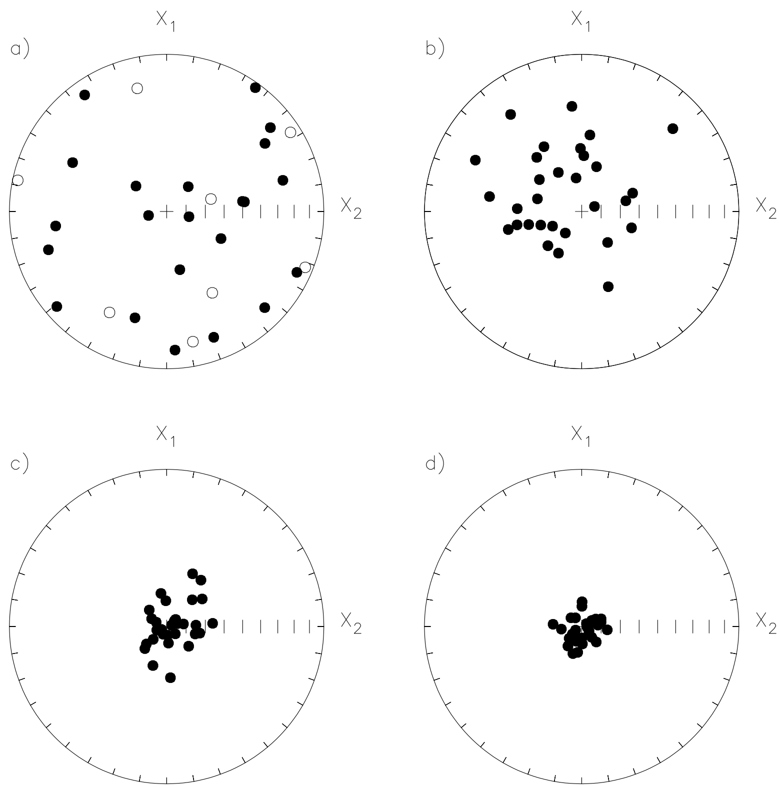

Figure 11.2: Four hypothetical data sets with decreasing scatter: a) is nearly uniformly distributed on the sphere, whereas d) is fairly well clustered. All data sets were drawn from Fisher distributions with vertical true directions.

Paleomagnetic directional data are subject to a number of factors that lead to scatter. These include:

1. uncertainty in the measurement caused by instrument noise or sample alignment errors,
2. uncertainties in sample orientation,
3. uncertainty in the orientation of the sampled rock unit,
4. variations among samples in the degree of removal of a secondary component,
5. uncertainty caused by the process of magnetization,
6. secular variation of the Earth's magnetic field, and
7. lightning strikes.

Some of these sources of scatter (e.g., items 1, 2 and perhaps 6 above) lead to a symmetric distribution about a mean direction. Other sources of scatter contribute to distributions that are wider in one direction than another. For example, in the extreme case, item four leads to a girdle distribution whereby directions are smeared along a great circle. In order to calculate mean directions with confidence limits, paleomagnetists rely heavily on the special statistics known as *Fisher statistics* (Fisher, 1953), which were developed for assessing dispersion of unit vectors on a sphere. In most instances, paleomagnetists assume a Fisher distribution for their data because the statistical treatment allows calculation of confidence intervals, comparison of mean directions, comparison of scatter, etc.

11.4 Parameter estimation

The Fisher probability density function (Fisher, 1953) is given by:

$$F = \frac{\kappa}{4\pi \sinh \kappa} \exp(\kappa \cos \alpha), \tag{11.1}$$

where α is the angle between the unit vector and the true direction and κ is a precision parameter such that as $\kappa \to \infty$, dispersion goes to zero.

Because the intensity of the magnetization has little to do with the validity of the measurement (except for very weak magnetizations), it is customary to assign unit length to all directions. The mean direction is calculated by first converting the individual directions (D_i, I_i) to cartesian coordinates (x_1, x_2, x_3) by the methods given in Lecture 2. The length of the resultant vector, R, is given by:

$$R^2 = (\sum_i x_{1i})^2 + (\sum_i x_{2i})^2 + (\sum_i x_{3i})^2, \tag{11.2}$$

and the cartesian coordinates of the mean direction are given by:

$$\bar{x}_1 = \frac{1}{R}(\sum_i x_{1i}); \quad \bar{x}_2 = \frac{1}{R}(\sum_i x_{2i}); \quad \bar{x}_3 = \frac{1}{R}(\sum_i x_{3i}). \tag{11.3}$$

The cartesian coordinates can, of course, be converted back to geomagnetic elements $(\bar{D}, \bar{I})$ by the familiar method described in Lecture 2.

The precision parameter for the Fisher distribution, κ, is estimated by

$$\kappa \simeq k = \frac{N-1}{N-R} \tag{11.4}$$

(where N is the number of data points). Using this estimate of κ, we estimate the circle of 95% confidence ($p = 0.05$) about the mean, α_{95}, by:

$$\alpha_{95} = \cos^{-1}[1 - \frac{N-R}{R}[(\frac{1}{p})^{\frac{1}{(N-1)}} - 1]]. \tag{11.5}$$

In the classic paleomagnetic literature, α_{95} was further approximated by:

$$\alpha'_{95} \simeq \frac{140}{\sqrt{kN}},$$

which is reliable for k larger than about 25 (see Tauxe et al., 1991).

Another useful parameter (introduced by Irving, 1964) is the so-called *circular standard deviation* (CSD), also sometimes called the angular standard deviation), which is approximated by:

$$CSD \simeq \frac{81}{\sqrt{k}},$$

which is the circle containing ~66% of the data.

If directions are converted to VGPs as outlined in Lecture 2, the transformation distorts a rotationally symmetric set of data into an elliptical distribution. The associated α_{95} may no longer be appropriate. Cox and Doell (1960) suggested the following for 95% confidence regions in VGPs. Ironically, it is more likely that the VGPs are spherically symmetric implying that most sets of directions are not!

$$dm = \alpha_{95} \frac{\cos\lambda}{\cos\bar{I}}$$

$$dp = \frac{1}{2}\alpha_{95}(1 + 3\sin^2\lambda), \tag{11.6}$$

where dm is the uncertainty in the paleomeridian (longitude), dp is the uncertainty in the paleoparallel (latitude), and λ is the site paleolatitude.

Two examples of Fisher distributions, one with a large degree of scatter (κ=5) and one that is relatively tightly clustered (κ=50) are shown in Figure 11.3. Also shown are the Fisher mean directions and α_{95}s for each data set.

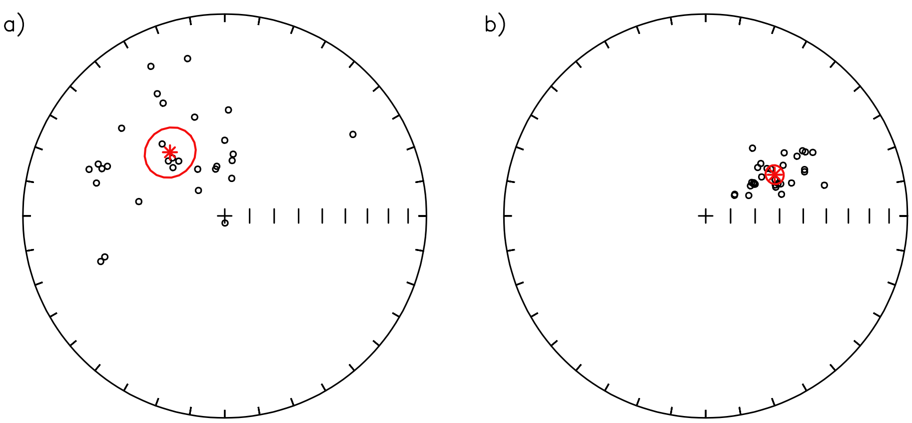

Figure 11.3: Two Fisher distributions: a) $\kappa = 5$, b) $\kappa = 50$. Mean directions are shown as asterisks, and α_{95}s as ellipses.

The Fisher distribution allows us to ask a number of questions about paleomagnetic data sets, such as:

1. Is a given set of directions random? This is the question that we ask when we perform a conglomerate test (Lecture 9).

2. Is the mean direction of a given (Fisherian) data set different from some known direction? This question comes up when we compare a given data set with, for example, the directions of the present or GAD field.

3. Are two (Fisherian) data sets different from each other? For example, are the normal directions and the antipodes of the reversed directions the same for a given data set?

4. If a given site has some samples that allow only the calculation of a best-fit plane and not a directed line, what is the site mean direction that combines the best-fit lines and planes (see Lecture 9)?

In the following discussion, we will briefly summarize ways of addressing these issues using Fisher techniques.

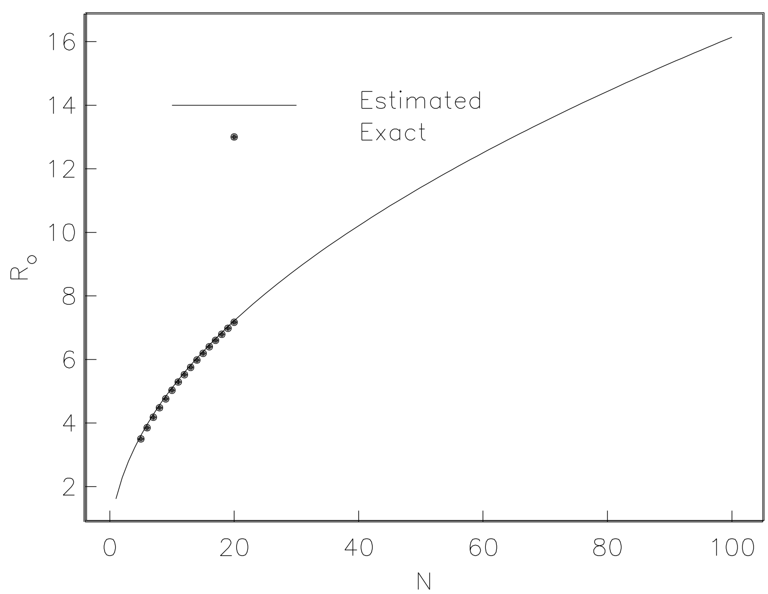

Figure 11.4: Values of R_o calculated by Equation 11.7 (line) and exactly (dots) for 95% level of confidence. Exact data are from Watson (1956).

11.5 Watson's test for randomness

Watson [1956] demonstrated how to test a given directional data set for randomness. His test relies on the calculation of R given by Equation 11.2. Because R is the length of the resultant vector, randomly directed vectors will have small values of R, while, for less scattered directions, R will approach N. Watson (1956) defined a parameter R_o that can be used for testing the randomness of a given data set. If the value of R exceeds R_o, the null hypothesis of total randomness can be rejected at a specified level of confidence. If R is less than R_o, randomness cannot be rejected. Watson calculated the value of R_o for a range of N for the 95% and 99% confidence levels. Watson (1956) also showed how to estimate R_o by:

$$R_o = \sqrt{7.815 \cdot N/3}. \tag{11.7}$$

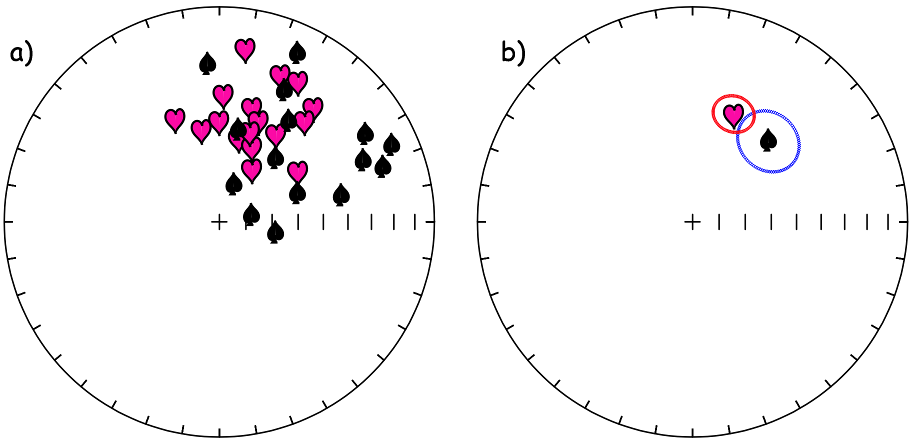

Figure 11.5: a) Equal area projections of declinations and inclinations of two hypothetical data sets. b) Fisher means and circles of confidence from the data sets in a).

We plot R_o versus N in Figure 11.4 using both the exact method (dots) and the estimation given by Equation 11.7. The estimation works well for $N > 10$, but is somewhat biased for smaller data sets. The critical values of R for $5 < N < 20$ from Watson (1956) are listed for convenience in Table D2.

The test for randomness is particularly useful for determining if, for example, the directions from a given site are randomly oriented (the data for the site should therefore be thrown out). Also, one can determine if directions from conglomerate clasts are randomly oriented in the conglomerate test (see Lecture 9).

11.6 Comparing known and estimated directions

The calculation of confidence regions for paleomagnetic data is largely motivated by a need to compare estimated directions with either a known direction (for example, the present field) or another estimated direction (for example, that expected for a particular paleopole, the present field or a GAD field). Comparison of a paleomagnetic data set with a given direction is straightforward using Fisher statistics. If the known test direction lies outside the confidence interval computed for the estimated direction, then the estimated and known directions are different at the specified confidence level.

11.7 Comparing two estimated directions

The case in which we are comparing two Fisher distributions can also be relatively straight forward. If the two confidence circles do not overlap, the two directions are different at the specified level of certainty. When one confidence region includes the mean of the other set of directions, the difference in directions is not significant.

The situtation becomes a little more tricky when the data sets are as shown in Figure 11.5a. The Fisher statistics for the two data sets are:

i	symbol	$\bar{D}$	$\bar{I}$	N	R	k	α_{95}
1	spades	43.3	47.1	20	17.9077	9.1	11.5
2	hearts	20.9	45.3	20	19.0908	20.9	7.3

As shown in the equal area projection in Figure 11.5b, the two α_{95}s overlap, but neither includes the mean of the other. This sort of "grey zone" case has been addressed by many workers. A particularly useful parameter (V_w) was proposed by Watson (1983; see Appendix for details).

V_w was posed as a test statistic that increases with increasing difference between the mean directions of the two data sets. Thus, the null hypothesis that two data sets have a common mean direction can be rejected if V_w exceeds some critical value which can be determined through what is called *Monte Carlo simulation*. The technique gets its name from famous gambling locale because we use randomly drawn samples ("cards") from specified distributions ("decks") to see what can be expected from chance. What we want to know is the probabilty that two data sets (hands of cards?) drawn from the same underlying distribution would have a given V_w statistic just from chance.

We proceed as follows:

1. Calculate the V_w statistic for the data sets. [The V_w for the two data sets shown in Figure 11.5a is 8.5.]

2. In order to determine the critical value for V_w, we draw two Fisher distributed data sets with dispersions of k_1 and k_2 and N_1, N_2, but having a common direction. You can try this "by hand", with the program **fisher** from the **pmag** distribution of programs (downloadable at sorcerer.ucsd.edu/software).

3. The calculate V_w for these simulated data sets.

4. Repeat the simulation some large number of times (say 1000). This defines the distribution of V_ws that you would get from chance by "sampling" a distributions with the same direction.

5. Sort the V_ws in order of increasing size. The critical value of V_w at the 95% level of confidence is the 950^{th} simulated V_w.

The V_ws simulated for two distributions with the same κ and N as our example data sets but drawn from distributions with the same mean are plotted in a histogram in Figure 11.6 with the bounds containing the lowermost 95% of the 1000 simulations shown as a dashed line. The value of 8.5, calculated for the data set is shown as a heavy vertical line and is clearly larger than 95% of the simulated populations which gives a critical value of 6.2. This simulation therefore supports the suggestion that the two data sets do not have a common mean at the 95% level of confidence.

This test can be applied to the two polarities in a given data collection to see if the they are antipodal. In this case, one would take the antipodes of one of the data sets before calculating V_w. This test is a Fisherian form of the *reversals test*.

11.8 Combining vectors and great circles

Consider the demagnetization data shown in Figure 11.7 for demagnetization data of various specimens from a certain site. The data from sample *tst1a* seem to reach some well defined direction and hover there. A mean direction for the last few demagnetization steps can be calculated using Fisher

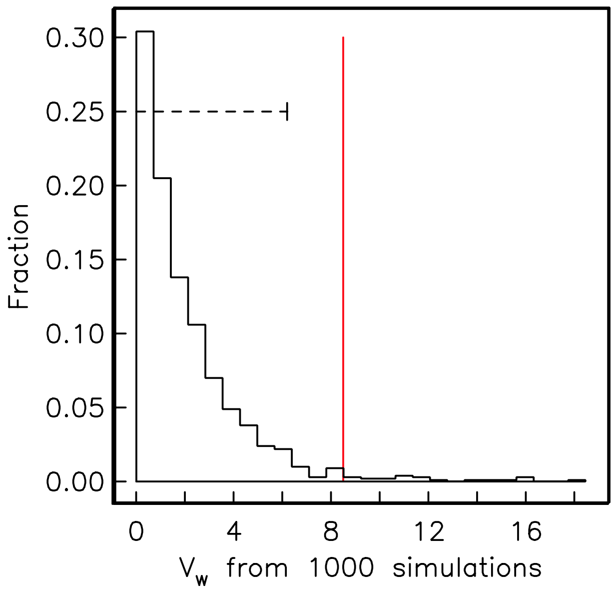

Figure 11.6: Distribution of V_w for simulated Fisher distributions with the same N and κ as the two shown in Figure 11.5. The dashed line includes the smalleds 95% of the V_ws calculated for the simulations. The heavy vertical line is the V_w calculated for the two data sets. According to this test, the two data sets do not have a common mean.

statistics. We can calculate a best-fit line from the data for sample *tst1b* (Figure 11.7b) using the principal component method of Kirschvink [1980] as outlined in Lecture 9. The data from *tst1c* track along a great circle path and can be used to calculate the pole to the best-fit plane calculated as in Lecture 9. McFadden and McElhinny (1988) described a method for estimating the mean direction and the α_{95} from sites that mix planes (great circles on an equal area projection) and directed lines (see Appendix). The key to their method is to find the direction within each plane that gives the tightest grouping of directions. Then "regular" fisher statistics can be applied.

11.9 Inclination only data

A different problem arises when only the inclination data are available as in the case of unoriented drill cores. Cores can be drilled and arrive at the surface in short, unoriented pieces. Specimens taken from such core material will be oriented with respect to the vertical, but the declination data are unknown. It is often desirable to estimate the true Fisher inclination of data set having only inclination data, but how to do this is not obvious. Consider the data in Figure 11.8. The true Fisher mean declination and inclination are shown by the asterisk. If we had only the inclination data and calculated a gaussian mean ($< I >$), the estimate would be too shallow as pointed out earlier.

Several investigators have addressed the issue of inclination-only data. McFadden and Reid (1982) developed a maximum likelihood estimate for the true inclination which works reasonably well. Their approach is outlined in the Appendix.

By comparing inclinations estimated using the McFadden-Reid technique with those calculated using the full vector data, it is clear that the method breaks down at high inclinations and high

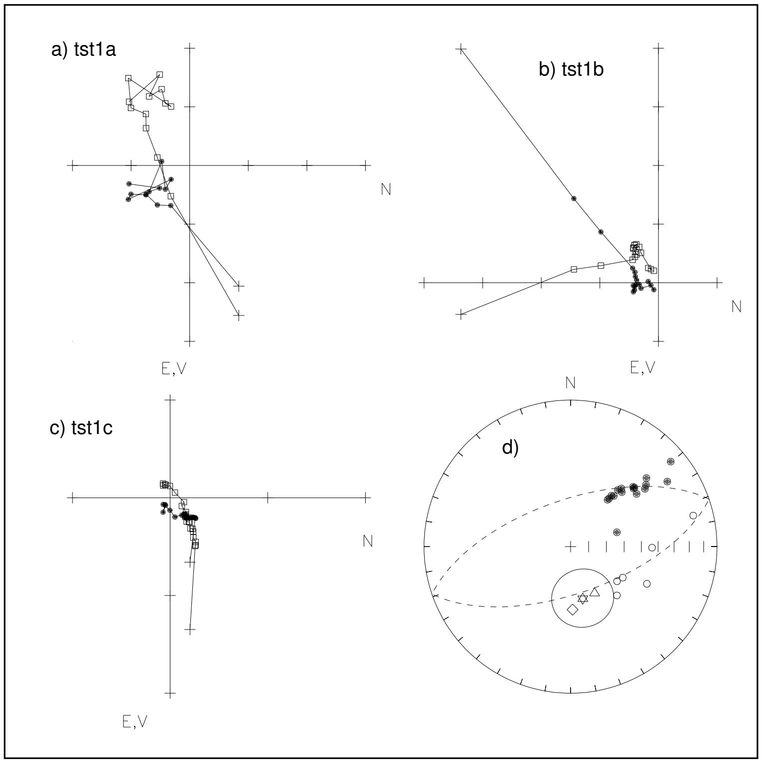

Figure 11.7: Examples of demagnetization data from a site whose mean is partially constrained by a great circle. The samples a)*tst1a*, b)*tst1b* and c) *tst1c* which are sibling samples from the same reversely magnetized site. The demagnetization data are plotted as orthogonal projections. The directional data from *tst1c* do not define a single component, but describe a great circle as shown in d). The sample *tst1b* allows calculation of a principal component whose direction is plotted as a diamond in d). Specimen *tst1a* has data that do not converge to the origin. A mean direction was calculated for this sample by standard Fisher statistics and is plotted as a triangle in d. The best-fit great circle and two directed lines allow a mean (star) and associated α_{95} to be calculated using the method of McFadden and McElhinny (1988).

scatter. It is also inappropriate for data sets that are not Fisher distributed!

11.10 Is a given data set Fisher distributed?

Clearly, the Fisher distribution allows powerful tests and this power lies behind the popularity of paleomagnetism in solving geologic problems. The problem is that these tests require that the data be Fisher distributed. How can we tell if a particular data set is Fisher distributed? What do we do if the data are not Fisher distributed? These questions are addressed in the rest of the lecture.

Let us now consider how to determine whether a given data set is Fisher distributed. There are actually many ways of doing this. There is a rather complete discussion of the problem in Fisher et al. (1987) and if you really want a complete treatment try the supplemental reading list. The quantile-quantile (Q-Q) method described by Fisher et al. (1987) is fairly intuitive and works well. We outline it briefly in the following.

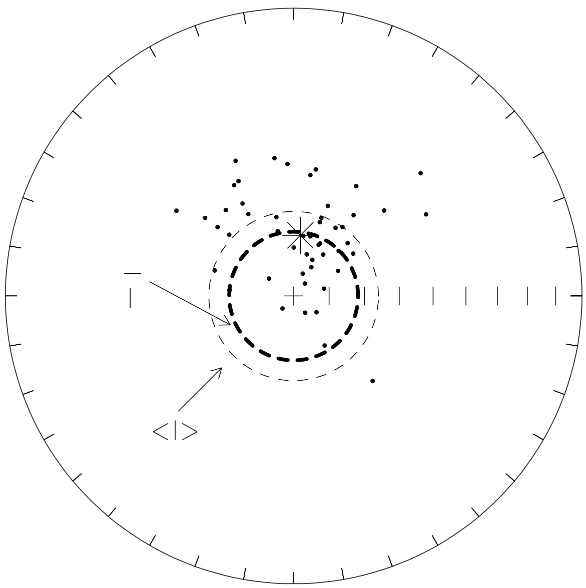

Figure 11.8: Directions drawn from a Fisher distribution with a near vertical true mean direction. The Fisher mean direction from the sample is shown by a asterisk. The Gaussian average inclination ($< I >$) is shallower than the Fisher mean $\bar{I}$.

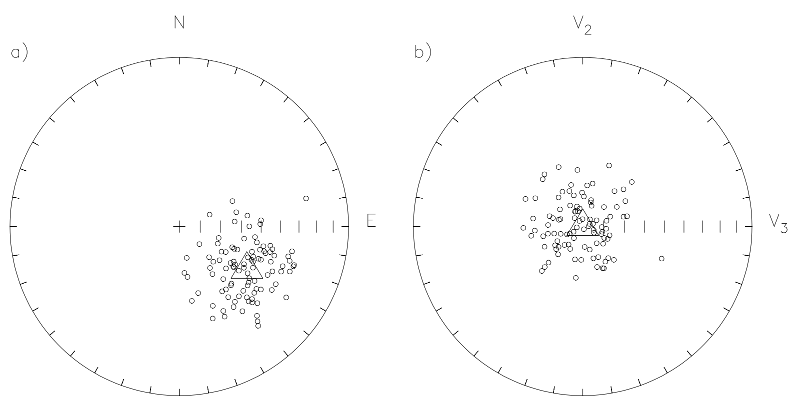

Figure 11.9: Transformation of coordinates from a) geographic to b) "data" coordinates. The direction of the principal eigenvector $\mathbf{V}_1$ is shown by the triangle in both plots.

The idea behind the Q-Q method is to exploit the fact that declinations in a Fisher distribution, when viewed along the mean, are spread around the clock evenly - there is a uniform distribution of declinations. Also, the inclinations (or rather the co-inclinations) are clustered close to the mean and the frequency dies off exponentially away from the mean direction.

Therefore, the first step in testing for Fisher-ness is to transpose the data such that the mean is the center of the distribution. You can think of this as rotating your head around to peer down the mean direction. On an equal area projection, the center of the diagram is the mean. In order to do this transformation, we first calculate the orientation matrix $\mathbf{T}$ of the data and the associated eigenvectors $\mathbf{V}_i$ and eigenvalues τ_i (Appendix to Lecture 9 - in case you haven't read it yet, do so NOW). Substituting the direction cosines relating the geographic coordinate system $\mathbf{X}$ to the coordinate system defined by $\mathbf{V}$, the eigenvectors, where $\mathbf{X}$ is the "old" and $\mathbf{V}$ is the "new" set of axes, we can transform the coordinate system for a set of data from "geographic" coordinates (Figure 11.9a) where the vertical axis is the center of the diagram, to the "data" coordinate system, (Figure 11.9b) where the principal eigenvector ($\mathbf{V}_1$) lies at the center of the diagram, after transformation into "data" coordinates.

Equation 11.1 for the Fisher distribution function suggests that declinations are symmetrically distributed about the mean. In "data" coordinates, this means that the declinations are uniformly distributed from $0 \rightarrow 360°$. Furthermore, the probability P of finding a direction of α away from the mean is exponential:

$$P = F \sin\alpha = \frac{\kappa}{4\pi \sinh\kappa} \exp\left(\kappa \cos\alpha\right) \sin\alpha. \tag{11.8}$$

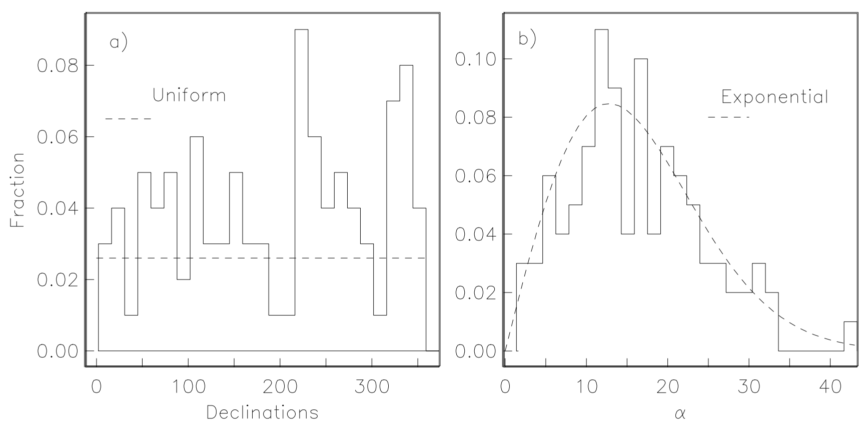

Figure 11.10: a) Declinations and b) co-inclinations (α) from Figure 11.9. Also shown are behaviors expected for D and I from a Fisher distribution, i.e., declinations are uniformly distributed while co-inclinations are exponentially distributed.

Let us compare the data from Figure 11.9 to the expected distributions for a Fisher distribution with $\kappa = 20$ (Figure 11.10). The data were generated using the program **fisher** which relies on the method outlined by Fisher et al. (1987), that draws directions from a Fisher distribution with a specified κ. We used a κ of 20, and it should come as no surprise that the data fit the expected distribution rather well. But how well is "well" and how can we tell when a data set *fails* to be fit by a Fisher distribution?

We wish to test whether the declinations are uniformly distributed and whether the inclinations are exponentially distributed as required by the Fisher distribution. Plots such as those shown in Figure 11.10 are not as helpful for this purpose as a plot known as a *Quantile-Quantile* (Q-Q) plot (see Fisher et al., 1987). In a Q-Q plot, the data are graphed against the value expected from a particular distribution. Data compatible with the chosen distribution plot along a line. The procedure for accomplishing this is given in the Appendix. In Figure 11.11a, we plot the declinations from Figure 11.9 (in data coordinates) against the values calculated assuming a uniform distribution and in Figure 11.11b, we plot the co-inclinations against those calculated using an exponential distribution. As expected, the data plot along lines and neither of the test statistics M_u nor M_e (see Appendix) exceed the critical values.

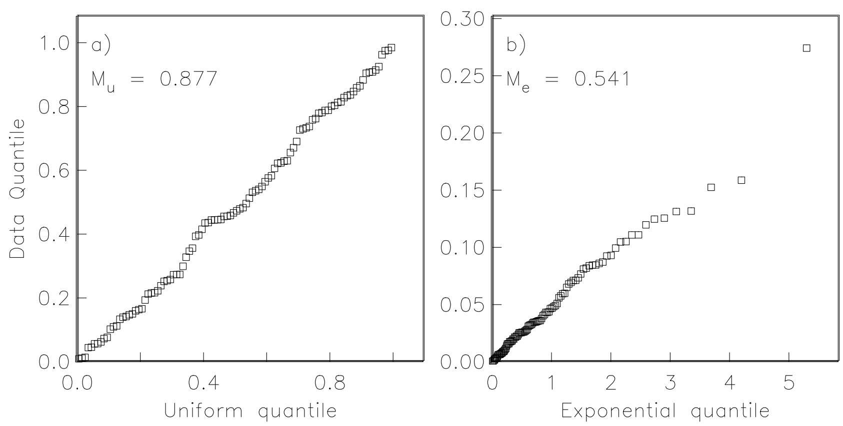

Figure 11.11: a) Quantile-quantile plot of declinations (in data coordinates) from Figure 9 plotted against an assumed uniform distribution. b) Same for inclinations plotted against an assumed exponential distribution. The data are Fisher distributed.

Appendix

A Calculation of Watson's V_w

1. Calculate R_i, and k_i where $i = 1, 2$ for the two data sets with N_1, N_2 samples using Equations 11.2 and 11.4.

2. Calculate $\bar{x}_{ij}$ (where $j = 1, 3$ for the three axes) using Equation 11.3.

3. Calculate $\bar{X}_{ij} = R_i \bar{x}_{ij}$.

4. Find the weighted means for the two data sets:

$$\hat{X}_j = \sum_i^2 k_j \bar{X}_{ij}$$

5. Calculate the weighted overall resultant vector R_w by

$$R_w = (\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2)^{\frac{1}{2}},$$

and the weighted sum S_w by,

$$S_w = \sum_i^2 k_i R_i.$$

6. Finally, Watson's V_w is defined as

$$V_w = 2(S_w - R_w).$$

B Combining lines and planes

1. Calculate M directed lines (2 in our case) and N great circles (1 in our case) using principal component analysis (see Lecture 9) or Fisher statistics.

2. Assume that the primary direction of magnetization for the samples with great circles lies somewhere along the great circle path (i.e., within the plane).

3. Assume that the set of M directed lines and N unknown directions are drawn from a Fisher distribution.

4. Iteratively search along the great circle paths for directions that maximize the resultant vector R for the $M + N$ directions.

5. Having found the set of N directions that lie along their respective great circles, estimate the mean direction using Equation 11.3 and κ as:

$$k = \frac{2M + N - 2}{2(M + N - R)},$$

The cone of 95% confidence about the mean is given by:

$$\cos \alpha_{95} = 1 - \frac{N' - 1}{kR}, [(\frac{1}{p})^{1/(N'-1)} - 1],$$

where $N' = M + N/2$ and $p = .02$

C Inclination only calculation

We wish to estimate the co-inclination ($\alpha = 90 - I$) of N Fisher distributed data (α_i), the declinations of which are unknown. We define the estimated value of α to be $\hat{\alpha}$. McFadden and Reid showed that $\hat{\alpha}$ is the solution of:

$$N \cos \hat{\alpha} + (\sin^2 \hat{\alpha} - \cos^2 \hat{\alpha}) \sum \cos \alpha_i - 2 \sin \hat{\alpha} \cos \hat{\alpha} \sum \alpha_i = 0,$$

which can be solved numerically.

They further define two parameters S and C as:

$$S = \sum \sin(\hat{\alpha} - \alpha_i),$$

$$C = \sum \cos(\hat{\alpha} - \alpha_i).$$

An unbiassed approximation for the Fisher parameter κ, k is given by:

$$k = \frac{N-1}{2(N-C)}.$$

The unbiased estimate $\hat{I}$ of the true inclination is:

$$\hat{I} = 90 - \hat{\alpha} + \frac{S}{C}.$$

Finally, the α_{95} is estimated by:

$$\cos\alpha_{95} = 1 - \frac{1}{2}(\frac{S}{C})^2 - \frac{f}{2Ck},$$

where f is the critical value taken from the F distribution (see Table A-1) with 1 and (N-1) degrees of freedom.

D Quantile-Quantile plots

In order to do this, we proceed as follows (Figure D1):

1. Sort the variable of interest ζ_i into ascending order so that ζ_1 is the smallest and ζ_N is the largest.

2. If the data are represented by the underlying density function as in Figure D1a, then the ζ_i's divide the curve into $(N+1)$ areas, A_i, the average value of which is $a = 1/(N+1)$. If we assume a form for the density function of ζ_i, we can calculate numbers z_i, that divide the theoretical distribution into areas a_i each having an area a (see Figure D1b).

3. An approximate test for whether the data ζ_i are fit by a given distribution is to plot the pairs of points (ζ_i, z_i), as shown in Figure D1c. If the assumed distribution is appropriate, the data will plot as a straight line.

4. The density function P is the distribution function F times the area, as mentioned before. The z_i are calculated as follows:

$$F(z_i) = (i - \tfrac{1}{2})/n, \text{where } i = 1, \ldots, n, \tag{D1}$$

so that:

$$z_i = F^{-1}((i - \tfrac{1}{2})/n), \text{where } i = 1, \ldots, n, \tag{D2}$$

and where F^{-1} is the inverse function to F. If the data are uniformly distributed (and constrained to lie between 0 and 1), then both $F(x)$ and $F^{-1}(x) = x$. For an exponential distribution $F(x) = 1 - e^{-x}$ and $F^{-1}(x) = -\ln(1-x)$.

D. QUANTILE-QUANTILE PLOTS

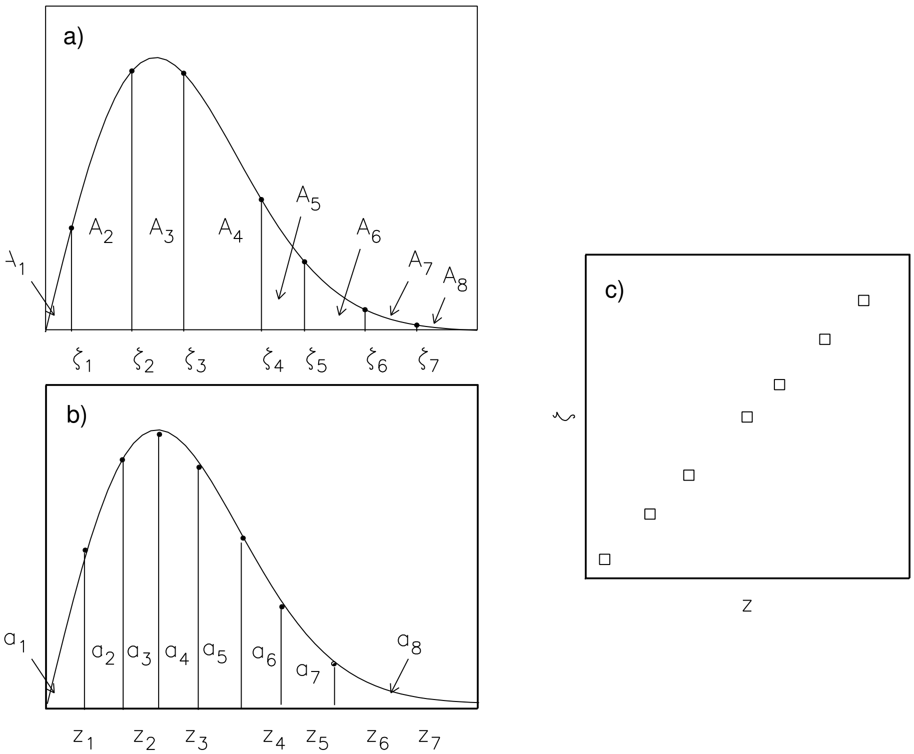

Figure D1: a) Illustration of how the sorted data ζ_i divide the density curve into areas A_i with an average area of $1/(N+1)$. b) The values of z_i which divide the density function into equal areas $a_i = 1/(N+1)$. c) Q-Q plot of z and ζ.

5. Finally, we can calculate parameters M_u and M_e which, when compared to critical values, allow rejection of the hypotheses of uniform and exponential distributions, respectively. To do this, we first calculate:

$$D_N^+ = \text{maximum}[\frac{i}{N} - F(x)], \tag{D3}$$

and

$$D_N^- = \text{maximum}[F(x) - \frac{(i-1)}{N}]. \tag{D4}$$

For a uniform distribution $F(x) = x$, so M_u is calculated by first calculating D_N^+ as the maximum of $[i/N - \zeta_i]$ and D_N^- as the maximum of $[\zeta_i - (i-1)/N]$. M_u is $D_N^+ + D_N^-$. A value of $M_u > 1.207$ (see Fisher et al., 1987) can be grounds for rejecting the hypothesis of uniformity at the 95% level of certainty. Similarly, $D_N^+ D_N^-$ can be calculated for the inclination (using $\zeta_i = 90 - I_i$) as the maximum of $[i/N - (1 - e^{-\zeta_i})]$ and maximum of $[(1-e^{-\zeta_i}) - (i-1)/N]$ respectively. $M_e = D_N^+ + D_N^-$. Values larger than 1.094 allow rejection of the exponential hypothesis. If either M_u or M_e exceed the critical values, the hypothesis of a Fisher distribution can be rejected.

Table D1: F-Tables for ν degrees of freedom.

ν	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19	19.16	19.25	19.3	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.1
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.5	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.1	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.2	3.09	3.01	2.95	2.9
12	4.75	3.89	3.49	3.26	3.11	3	2.91	2.85	2.8
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.6	3.74	3.34	3.11	2.96	2.85	2.76	2.7	2.65
15	4.54	3.68	3.29	3.06	2.9	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.2	2.96	2.81	2.7	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.9	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.1	2.87	2.71	2.6	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.3	3.44	3.05	2.82	2.66	2.55	2.46	2.4	2.34
23	4.28	3.42	3.03	2.8	2.64	2.53	2.44	2.37	2.32
24	4.26	3.4	3.01	2.78	2.62	2.51	2.42	2.36	2.3
25	4.24	3.39	2.99	2.76	2.6	2.49	2.4	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.2	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.7	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4	3.15	2.76	2.53	2.37	2.25	2.17	2.1	2.04
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96

Table 1 - continued.

ν	10	12	15	24	30	40	60	120	Inf
1	241.88	243.90	245.95	249.05	250.1	251.14	252.2	253.25	254.32
2	19.4	19.41	19.43	19.45	19.46	19.47	19.48	19.49	19.5
3	8.79	8.74	8.7	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.53	4.5	4.46	4.43	4.4	4.37
6	4.06	4	3.94	3.84	3.81	3.77	3.74	3.7	3.67
7	3.64	3.57	3.51	3.41	3.38	3.34	3.3	3.27	3.23
8	3.35	3.28	3.22	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.9	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.74	2.7	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.61	2.57	2.53	2.49	2.45	2.4
12	2.75	2.69	2.62	2.51	2.47	2.43	2.38	2.34	2.3
13	2.67	2.6	2.53	2.42	2.38	2.34	2.3	2.25	2.21
14	2.6	2.53	2.46	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.4	2.29	2.25	2.2	2.16	2.11	2.07
16	2.49	2.42	2.35	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.19	2.15	2.1	2.06	2.01	1.96
18	2.41	2.34	2.27	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.2	2.08	2.04	1.99	1.95	1.9	1.84
21	2.32	2.25	2.18	2.05	2.01	1.96	1.92	1.87	1.81
22	2.3	2.23	2.15	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.2	2.13	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.95	1.9	1.85	1.8	1.75	1.69
27	2.2	2.13	2.06	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.1	2.03	1.9	1.85	1.81	1.75	1.7	1.64
30	2.16	2.09	2.01	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2	1.92	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.7	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.61	1.55	1.5	1.43	1.35	1.25

Table D2: Critical values of R_o for a random distribution [Watson, 1956.]

N	95%	99%	N	95%	99%
5	3.50	4.02	13	5.75	6.84
6	3.85	4.48	14	5.98	7.11
7	4.18	4.89	15	6.19	7.36
8	4.48	5.26	16	6.40	7.60
9	4.76	5.61	17	6.60	7.84
10	5.03	5.94	18	6.79	8.08
11	5.29	6.25	19	6.98	8.33
12	5.52	6.55	20	7.17	8.55

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